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CONSERVATION LAWS FOR MATERIALS EXHIBITING POWER-LAW CREEP

T. J. DELPH
Lehigh University, Bethlehem, PA 18015, U.S.A.

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Abstract—Conservation laws are derived from materials whose constitutive behavior is characterized by power-law creep with elastic strains. This is accomplished by formulating an adjoint variational principle which has as its Euler-Lagrange equations the governing equations as well as a set of adjoint equations involving adjoint variables. Conservation laws are then derived by an application of Noether's theorem to the variational principle. The results are analogous to those obtained in linear elasticity, in that conservation laws are shown to arise from translations of spatial and temporal coordinates, rigid-body rotations, and self-similar scalings. A path-independent integral formulation of one of the conservation laws, valid under special circumstances, is derived.

INTRODUCTION

The derivation and application of conservation laws applicable to various types of material behavior has been a topic of recent interest in mechanics [1-7]. Conservation laws, by definition, possess the general mathematical form

$$\partial A_k/\partial x_k + \partial A_4/\partial t = 0 \tag{1}$$

where x_k ($K = 1, 2, 3$) are the spatial coordinates, t denotes time, and the A_k are functions of the independent and dependent variables for the problem. Equation (1) may be converted to an equivalent integral formulation by integrating over a spatial volume V and applying the divergence theorem to obtain

$$\int_S A_k \nu_k \,\mathrm{d}S + \int_V (\partial A_4/\partial t)\,\mathrm{d}V = 0 \tag{2}$$

where S is the bounding surface of V having unit normal vector ν_k. Equation (2) may be recognized to have the form generally associated with a conservation law: an equation expressing a balance between surface flux terms and the time rate of change of some conserved quantity (A_4 in this case) throughout the volume V. If the problem is time-independent, we obtain only the surface integral term in eqn (2). In two dimensions, this can lead under proper circumstances to a path-independent integral form, a form which has applications in problems in fracture mechanics.

In 1918, Emmy Noether [8] developed a useful and quite straightforward means of deriving conservation laws for a physical system whose governing equations are derivable from a variational principle. The resulting theorem essentially states that there exists a conservation law corresponding to every infinitesimal transformation of the independent and dependent variables which leaves the Lagrangian density unaltered to first order. Noether's theorem was first applied to static elasticity by Günther [1] and later independently by Knowles and Sternberg [2], while the dynamic case was treated by Fletcher [3]. Recently Delph [4] has noted the existence of conservation laws in elasticity generated by divergence transformations.

Almost all of the applications to date of these conservation laws have been to nonlinear fracture problems and have utilized the well-known J-integral conservation law, first developed and applied in this regard by Rice [9]. Since Rice's initial work, there has developed a voluminous literature dealing with various applications of the J-integral to a variety of nonlinear material behaviors. The J-integral is strictly valid only for linear or nonlinear elasticity, but may be extended to the deformation theory of plasticity if no unloading is allowed, or to power-law creep if elastic strains are excluded. However it has been widely applied in an approximate sense to cases where these restrictions are violated.

The purpose of the present work is to explore the possibility of deriving conservation laws for nonlinear time-dependent material behavior. In particular, we wish to consider conservation laws satisfied by bodies whose constitutive behavior is that of power-law creep with elastic strains.

VARIATIONAL PRINCIPLE

Under quasi-static conditions, the appropriate field equations are the equilibrium equation

$$\sigma_{ij,j} = 0 \tag{3}$$

and the constitutive relation

$$(\dot{u}_{i,j} + \dot{u}_{j,i})/2 = C_{ijkl}\dot{\sigma}_{kl} + (3/2)B\sigma_e^{\,n-1}\sigma'_{ij} \tag{4}$$

where σ_{ij} is the stress tensor, $\dot{u}_i$ the displacement rate, c_{ijkl} the elastic constants, σ'_{ij} the deviatoric stress tensor, and $\sigma_e^2 = (3/2)\sigma'_{ij}\,\sigma'_{ij}$. The quantities B and n are material constants. Here $(\dot{\ }) \equiv \partial/\partial t$ and $(\ )_{,j} \equiv \partial/\partial x_j$.

If the elastic strain rate term is taken to vanish in eqn (4), then, as noted by Goldman and Hutchinson [10], equations (3) and (4) become essentially time-independent. The J-integral conservation law, as well as other of the Günther–Knowles–Sternberg conservation laws, then becomes strictly valid. However the inclusion of the elastic strain term renders eqn (3) and (4) truly time-dependent and the aforementioned conservation laws then lose their validity.

In order to apply Noether's theorem, eqns (3) and (4) should be derivable from a variational principle. This, however, can easily be shown not to be possible. The conditions which must be satisfied in order that a given set of partial differential equations represent the Euler–Lagrange equations for some variational principle are given by Tonti [11] and involve requirements somewhat similar to those for formal self-adjointness in the case of ordinary differential equations. It is not difficult to verify that the inclusion of the elastic strains in eqn (4) results in the violation of those requirements, and hence that there exists no variational principle for material behavior characterized by power-law creep with elastic strains.

One technique for circumventing this difficulty is the formulation of an adjoint variational principle along the lines suggested by Morse and Feshbach for the linear diffusion equation [12]. Here a set of adjoint variables are introduced corresponding to the set of dependent variables in the original problem. A functional is then constructed which has as its Euler–Lagrange equations both the original partial differential equations and a set of adjoint partial differential equations involving both the original and adjoint variables. By virtue of the method by which the functional is constructed, the adjoint equations will always be linear in the adjoint variables. In the area of mechanics, this technique has been used by Tasi and Herrmann [13] to obtain a variational principle for linear thermoelasticity, and later employed by Herrmann [5] to derive a set of associated conservation laws. It has also been used by Finlayson [14] to construct a variational principle for the Navier–Stokes equations.

In eqns (3) and (4), the dependent variables are the displacement rate $\dot{u}_i$ and the stress tensor σ_{ij}. If we take $\dot{v}_i$ and Σ_{ij} to be the corresponding adjoint variables (Σ_{ij} is assumed to be symmetric), we may define a Lagrangian density L by

$$L = \dot{v}_i\sigma_{ij,j} + \Sigma_{ij}(-\dot{u}_{i,j} + C_{ijkl}\dot{\sigma}_{kl} + (3/2)B\sigma_e^{\,n-1}\sigma'_{ij}). \tag{5}$$

As boundary conditions for the adjoint variables, we prescribe the adjoint displacement rate to be $\dot{\bar{v}}_i$ over some portion S_v of the boundary and the adjoint stresses to be $\bar{\Sigma}_{ij}$ over the remaining portion S_Σ. We now define a functional I by

$$I = \int_{t_0}^{t_1}\int_V L\,\mathrm{d}V\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{S_v} \dot{\bar{v}}_i\sigma_{ij}\nu_j\mathrm{d}S\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{S_\Sigma} \bar{\Sigma}_{ij}\dot{u}_j\nu_i\,\mathrm{d}S\,\mathrm{d}t \tag{6}$$

and require the vanishing of the first variation of I. After an integration by parts with respect to time and an applicaiton of the divergence theorem, we obtain

$$\int_{t_0}^{t}\int_V \sigma_{ij,j}\,\delta\dot{v}_i\,\mathrm{d}V\mathrm{d}t + \int_{t_0}^{t_1}\int_V (-\dot{u}_{i,j} + C_{ijkl}\,\dot{\sigma}_{kl} + (3/2)\,B\sigma_e^{n-1}\sigma'_{ij})\delta\Sigma_{ij}\,\mathrm{d}V\,\mathrm{d}t$$

$$-\int_{t_0}^{t}\Sigma_{ij,j}\,\delta\dot{u}_i\,\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_V\{-\dot{v}_{i,j} - C_{ijkl}\dot{\Sigma}_{kl}$$

$$+ (3/2)B\sigma_e^{n-1}[(3/2)(n-1)\sigma'_{ij}\sigma'_{kl}/\sigma_e^2 + \delta_{ik}\delta_{jl} - \delta_{kl}\delta_{ij}/3]\Sigma_{kl}\}\delta\sigma_{ij}\,\mathrm{d}V\,\mathrm{d}t$$

$$+\int_V C_{ijkl}\,\Sigma_{ij}\delta\sigma_{kl}\,\mathrm{d}V + \int_{t_0}^{t_1}\int_{S_t}(\dot{v}_i - \bar{\dot{v}}_i)\delta\sigma_{ij}\nu_j\,\mathrm{d}S - \int_{t_0}^{t_1}\int_{S_\Sigma}(\Sigma_{ij} - \bar{\Sigma}_{ij})\delta\dot{u}_j\nu_i\,\mathrm{d}S\,\mathrm{d}t = 0. \tag{7}$$

The Euler–Lagrange equations corresponding to the functional I are now seen to be eqns (3) and (4) and the adjoint equations

$$\Sigma_{ij,j} = 0 \tag{8}$$

$$(\dot{v}_{i,j} + \dot{v}_{j,i})/2 = -C_{ijkl}\,\dot{\Sigma}_{kl} + (3/2)B\sigma_e^{n-1}[(3/2)(n-1)\sigma'_{ij}\,\sigma_{kl}'/\sigma_e^2$$

$$+\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{ij}/3]\Sigma_{kl}. \tag{9}$$

The satisfaction of the Euler–Lagrange equations with the indicated boundary conditions on $\dot{v}_i$ and Σ_{ij}, along with the requirement that $\delta\sigma_{kl} = 0$ at $t = t_0, t_1$, can be seen to render the functional I stationary. As noted previously, the adjoint eqns (8) and (9) are linear in the adjoint variables, though the coefficients may possibly be nonlinear functions of the regular variables.

CONSERVATION LAWS

We now turn our attention to possible conservation laws derivable from the variational principle whose functional is given by eqn (6). In accordance with Noether's theorem, we thus want to investigate classes of infinitesimal transformations of the independent and dependent variables of the form

$$\bar{x}_i = x_i + \delta x_i;\ \bar{t} = t + \delta t$$

$$\bar{\dot{u}}_i = \dot{u}_i + \delta\dot{u}_i;\ \bar{\sigma}_{ij} = \sigma_{ij} + \delta\sigma_{ij} \tag{10}$$

$$\bar{\dot{v}}_i = \dot{v}_i + \delta\dot{v}_i;\ \bar{\Sigma}_{ij} = \Sigma_{ij} + \delta\Sigma_{ij}.$$

Noether's theorem asserts that there exists a conservation law corresponding to every group of transformations (10) which leave the action integral for the variational principle invariant to first order in the infinitesimal quantities, i.e.

$$\int_{t_0}^{t_1}\int_V L\,\mathrm{d}V\,\mathrm{d}t = \int_{\bar{t}_0}^{\bar{t}_1}\int_{\bar{V}}\bar{L}\,\mathrm{d}\bar{V}\,\mathrm{d}\bar{t}. \tag{11}$$

In the ensuing treatment we will assume that $L = \bar{L}$, that is, we will ignore the possibility of divergence transformations on the Lagrangian density function such as has been considered in [4]. With this assumption, the relationship which the transformations given by eqn (10) must satisfy in order that eqn (11) hold in the present case is, from Ref. [15]

$$\delta\dot{v}_i(\partial L/\partial\dot{v}_i) + \delta\Sigma_{ij}(\partial L/\partial\Sigma_{ij}) + \delta\sigma_{ij}(\partial L/\partial\sigma_{ij}) + \delta\dot{u}_{i,j}(\partial L/\partial\dot{u}_{i,j})$$

$$+\delta\sigma_{ij,k}(\partial L/\partial\sigma_{ij,k}) + \delta\dot{\sigma}_{ij})\,(\partial L/\partial\dot{\sigma}_{ij}) + [(\partial\delta x_k/\partial x_k) + \partial\delta t/\partial t]L = 0. \tag{12}$$

Here

$$\delta \dot{u}_{i,j} = \mathrm{d}(\delta \dot{u}_i)/\mathrm{d}x_j - \dot{u}_{i,k}\, \partial(\delta x_k)/\partial x_j$$

$$\delta \sigma_{ij,k} = \mathrm{d}(\delta \sigma_{ij})/\mathrm{d}x_k - \sigma_{ij,l}\, \partial(\delta x_l)/\partial x_k \tag{13}$$

$$\delta \dot{\sigma}_{ij} = \mathrm{d}(\delta \sigma_{ij})/\mathrm{d}t - \dot{\sigma}_{ij}\, \partial(\delta t)/\partial t.$$

In eqn (13) and subsequently, we use the total derivative notation to indicate differentiation according to the chain rule, i.e.

$$\mathrm{d}(\delta \dot{u}_i)/\mathrm{d}x_j = \partial(\delta \dot{u}_j)/\partial x_j + (\partial(\delta \dot{u}_i)/\partial \dot{u}_k)\dot{u}_{k,j} + \ldots \tag{14}$$

whereas the partial derivative notation indicates differentiation with respect to the explicitly appearing variable only.

For every group of transformations which satisfies eqn (13), and hence eqn (11), Noether's theorem shows that there exists a conservation law which, in the present case, has the form [15]

$$(\mathrm{d}/\mathrm{d}x_k)\{[L\delta_{lk} - (\partial L/\partial \dot{u}_{i,k})\dot{u}_{i,l} - (\partial L/\partial \sigma_{ij,k})\sigma_{ij,l}]\delta x_l - [(\partial L/\partial \sigma_{ij,k})\dot{\sigma}_{ij} + (\partial L/\partial \dot{u}_{i,k})\dot{u}_i]\delta t$$

$$+ (\partial L/\partial \dot{u}_{i,k})\delta u_i + (\partial L/\partial \sigma_{ij,k})\delta \sigma_{ij}\} + (\mathrm{d}/\mathrm{d}t)\{[L - (\partial L/\partial \dot{\sigma}_{ij})\dot{\sigma}_{ij}]\delta t - (\partial L/\partial \dot{\sigma}_{ij})\sigma_{ij,l}\delta x_l + (\partial L/\partial \dot{\sigma}_{ij})\delta \dot{\sigma}_{ij}\} = 0. \tag{15}$$

We now wish to examine some particular groups of transformations.

(a) *Transition of spatial coordinates*

Since the Lagrangian density L given by eqn (5) does not contain the independent variables explicitly, a spatial translation obviously represents an invariant transformation. We take

$$\delta x_i = \epsilon \delta_{ni}\ (n \text{ fixed});\ \delta t = \delta \dot{u}_i = \delta \sigma_{ij} = 0 \tag{16}$$

$$\delta \dot{v}_i = \delta \Sigma_{ij} = 0$$

which corresponds to an infinitesimal translation of magnitude ϵ along the n coordinate axis. The associated conservation law from eqn (15) is

$$(\mathrm{d}/\mathrm{d}x_k)[L\delta_{kn} + \Sigma_{ik}\dot{u}_{i,n} - \dot{v}_i\sigma_{ik,n}] - (\mathrm{d}/\mathrm{d}t)(C_{ijkl}\Sigma_{kl}\sigma_{ij,n}) = 0. \tag{17}$$

A similar transformation leads to the J-integral conservation law in linear elasticity [1-3].

(b) *Translation along the time axis*

Here we consider an infinitesimal shift of magnitude ϵ along the time axis

$$\delta t = \epsilon;\ \delta x_i = \delta \dot{u}_i = \delta \sigma_{ij} = \delta \dot{v}_i = \delta \Sigma_{ij} = 0$$

for which the conservation law is

$$-(\mathrm{d}/\mathrm{d}x_k)(\dot{v}_i\dot{\sigma}_{ik} - \Sigma_{ik}\ddot{u}_i) + (\mathrm{d}/\mathrm{d}t)(L - C_{ijkl}\dot{\sigma}_{ij}\Sigma_{kl}) = 0. \tag{18}$$

In the case of linear elasticity, this transformation leads to the law expressing conservation of mechanical energy [3]. A similar interpretation seems to be indicated here since, as noted by Morse and Feshbach [12], the adjoint system is constructed so as to absorb energy at the same rate as which energy is dissipated in the regular system, leading to an overall conservation of energy.

(c) *Rotation of spatial coordinates*

Here we consider an infinitesimal rotation θ about the n coordinate axis given by

$$\delta x_i = \theta\epsilon_{nij}x_j;\ \delta\dot{u}_i = \theta\epsilon_{nij}\dot{u}_j;\ \delta\dot{v}_i = \theta\epsilon_{nij}\dot{v}_j;\ \delta t = 0$$

$$\delta\sigma_{ij} = \theta(\delta_{jm}\epsilon_{nil} + \delta_{il}\epsilon_{njm})\sigma_{lm};\ \delta\Sigma_{ij} = \theta(\delta_{jm}\epsilon_{nil} + \delta_{il}\epsilon_{njm})\Sigma_{lm}. \tag{19}$$

Here, as in the case of linear elasticity [1–3], invariance under the transformations given by eqn (19) requires that the material behavior be isotropic. The inelastic strain rate term as given in eqn (4) already embodies this requirement, so we must require here that the elastic constants C_{ijkl} be the isotropic constants. If this is done, it can be verified with a little effort that the transformations (19) satisfy eqn (13). The associated conservation law is then found to be

$$(\mathrm{d}/\mathrm{d}x_k)[(L\delta_{lk} + \Sigma_{ik}\dot{u}_{i,l} - \dot{v}_i\sigma_{ik,l})\epsilon_{nlj}x_j - \Sigma_{ik}\epsilon_{nij}\dot{u}_j + \dot{v}_i(\delta_{im}\epsilon_{nkl} + \delta_{kl}\epsilon_{nim})\sigma_{lm}] + (\mathrm{d}/\mathrm{d}t)[-C_{ijkl}\Sigma_{kl}\sigma_{ij,m}\epsilon_{nmj}x_j$$

$$+ C_{ijkl}\Sigma_{kl}(\delta_{jm}\epsilon_{nip} + \delta_{ip}\epsilon_{njm})\sigma_{mp}] = 0. \tag{20}$$

(d) *Scaling of coordinates*

Finally, we consider the scaling transformations given by

$$\delta x_l = \epsilon q x_l;\ \delta t = \epsilon(1-n)t$$

$$\delta\dot{u}_i = \epsilon(n+q)\dot{u}_i;\ \delta\dot{v}_i = \epsilon(n+q)\dot{v}_i \tag{21}$$

$$\delta\sigma_{ij} = \epsilon\sigma_{ij};\ \delta\Sigma_{ij} = \epsilon\Sigma_{ij}$$

where q is a free parameter having arbitrary value. These transformations can be shown to satisfy the invariance requirement given by eqn (13), and lead to the conservation law

$$(\mathrm{d}/\mathrm{d}x_k)[Lqx_k + q\Sigma_{ik}\dot{u}_{i,l}x_l - \dot{v}_j\sigma_{kj,l}qx_l - (1-n)\dot{v}_j\sigma_{kj}t + (1-n)\Sigma_{ik}\ddot{u}_i t - (n+q)\Sigma_{ik}\dot{u}_i + \dot{v}_j\sigma_{kl}] +$$

$$+ (\mathrm{d}/\mathrm{d}t)[(1-n)Lt - (1-n)C_{ijkl}\dot{\sigma}_{ij}\Sigma_{kl}t - qC_{ijkl}\sigma_{ij,n}\Sigma_{kl}x_n + C_{ijkl}\sigma_{ij}\Sigma_{kl}] = 0. \tag{22}$$

The transformations (21) are directly related to a family of one-parameter self-similar scaling solutions of eqns (3) and (4) and represent a generalization of a set of self-similar solutions which were apparently first given by Riedel [16]. The role of self-similar solutions in the generation of conservation laws has been discussed in a general context by Edelen [17].

DISCUSSION

In the foregoing we have derived sets of conservation laws valid for time-dependent material behavior characterized by power law creep with elastic strains. These conservation laws are somewhat analogous to those of linear elasticity [1–3], being based on invariance of the action integral under space and time translations, rigid body rotations, and self-similar scalings. As noted, we have excluded from consideration the possibility of divergence transformations, such as those discussed in [4] for the case of linear elasticity.

Because the governing eqns (3) and (4) are not derivable from a variational principle, we have chosen to construct an adjoint variational principle to which Noether's theorem could be applied. Other approaches to deriving conservation laws are possible which we shall discuss shortly. The present method is advantageous in that it yields conservation laws of the classical form given by eqns (1) and (2) in a straightforward fashion. It has, however, a major drawback in that it is necessary to introduce a set of new dependent variables adjoint to the original ones. This effectively doubles the size of the problem which must be considered. This difficulty is somewhat mitigated by the fact that the adjoint equations, eqns (8) and (9) in the present case, are always linear in the adjoint variables by virtue of the method by which they are constructed. This makes them somewhat easier to handle than the original nonlinear equations. However there is no doubt that introduction of the adjoint variables adds an extra measure of complexity to the problem.

From the point of view of applications, another difficulty arises from the form of the conservation laws. Since the material behavior is assumed to be time-dependent, we always obtain a term representing the time derivative of some quantity, as in the second term of eqn (1). In the integral formulation given by eqn (2), this term becomes a volume integral. The presence of this term makes it quite difficult, except in special cases, to express the conservation law as a "path-independent integral", a formulation which has been the basis of almost all of the applications of conservation laws to date in solid mechanics.

However, as Eshelby has noted [18], it is possible to introduce a Galliean transformation of coordinates which will take the volume integral term in eqn (2) into a surface integral if the problem can be assumed to be rendered time-invariant under such a transformation. A particular example is that of steady-state crack propagation in an unbounded body. To an observer translating with the crack tip, the stress and deformation fields about the crack appear to be time-independent. In order to be more definite, consider the case, recently studied by Hui and Riedel [19], of two-dimensional quasi-static crack propagation along the x_1 coordinate axis. Let $\dot{a}$ be the constant crack propagation rate. If we then consider the origin of coordinates to translate with the crack tip, under steady state conditions we have

$$(\partial/\partial t) = -\dot{a}(\partial/\partial x_1). \tag{24}$$

Introducing eqn (24) into eqn (17), and setting $n = 1$, there results

$$(\mathrm{d}/\mathrm{d}x_k)[L\delta_{k1} + \Sigma_{ik}\dot{u}_{i,1} - \dot{v}_i\sigma_{ik,1} + \dot{a}C_{ijmn}\Sigma_{mn}\sigma_{ij,1}\delta_{k1}] = 0. \tag{25}$$

We now integrate eqn (25) over some two-dimensional area A and apply the divergence theorem. Noting that the field eqns (3) and (4) lead to the vanishing of L over A and the bounding contour S, we obtain

$$\oint_S (\Sigma_{ik}\dot{u}_{i,1}\nu_k - \dot{v}_i\sigma_{ik,1}\nu_k + \dot{a}C_{ijmn}\Sigma_{mn}\sigma_{ij,1}\nu_1)\,\mathrm{d}S = 0 \tag{26}$$

where ν_i are the components of the unit normal vector to S. If we impose traction-free boundary conditions over the crack face and note that $\nu_1 = 0$ here as well, then the integrand of eqn (26) represents a "path-independent integral" for the case of steady-state creep crack propagation in an unbounded body, and may find possible applications in the analysis of this problem.

Finally, we discuss briefly other possible approaches to the derivation of conservation laws which do not depend on the use of adjoint variational principles. One such approach is through the use of variational principles based on convolution products, such as have been given for linear viscoelasticity by Gurtin [20] and more recently by Francfort and Herrmann [7] for linear thermoelasticity. As noted, these variational principles make use of convolution products in the time variable, and have been used by Francfort and Herrmann [7] as a basis for the derivation of conservation laws. Additionally, a conservation law for linear viscoelasticity derived by Nilsson [21] appears to be related to one of the variational principles of Gurtin. This entire approach, however, seems to depend upon the linearity of the governing equations and hence is of dubious applicability to nonlinear material behavior of the sort exemplified by eqn (4).

A more promising approach is through the use of various techniques involving the use of the exterior calculus [17, 22, 23]. These techniques are attractive in that they do not require the existence of a variational principle, as does Noether's theorem, but rather work directly with the governing equations. In particular, Edelen [17, 23] has shown how the isovector fields for the equations of linear elasticity may be used to generate the corresponding conservation laws. The application of isovector techniques to eqns (3) and (4) will be discussed in a subsequent work.

Finally we note the existence of a number of so-called restricted variational principles which lead to eqns (3) and (4), provided that certain of the first variations of the dependent variables are arbitrarily required to vanish. Several of these are discussed in the book by Rabotnov [24]. These, however, are not variational principles in the true sense of the term, and because of the artificial restrictions involved in their construction, cannot be used in conjunction with Noether's theorem as the basis for conservation laws.

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